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# **Extensions of modules for $SL(2, K)$**

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*Proposed running head:*

Extensions of modules for  $SL(2, K)$

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In this paper, we consider the induced modules  $\nabla$  and the Weyl modules  $\Delta$  for the algebraic group  $G = SL(2, K)$  where  $K$  is an algebraically closed field of characteristic  $p > 0$ . We determine the  $G$ -modules  $H^i(G_1, \nabla(s) \otimes \nabla(t))$  for all  $i \geq 0$ , where  $G_1$  is the first Frobenius kernel of  $G$ . We then use it to find the  $\text{Ext}^1$ -spaces between twisted tensor products of Weyl modules and induced modules for  $G$ . Moreover, we describe explicitly the non-split extensions corresponding to  $\nabla$ 's.

*Key words:* special linear group, symmetric powers, decomposition matrix.

# Introduction

In the theory of highest weight categories, the classes of modules  $\nabla$  and  $\Delta$  are of central interest. In particular, twisted tensor products of these modules occur as important subquotients of  $\nabla$  and  $\Delta$  (see [12] and [13]).

Here we consider these modules for the group  $G = SL(2, K)$ , the special linear group of dimension 2 over an algebraically closed field  $K$  of characteristic  $p > 0$ . Suppose that  $F : G \longrightarrow G$  is the corresponding Frobenius morphism and let  $G_1$  denote the first Frobenius kernel of  $G$ . If  $V$  is a  $G$ -module then we denote by  $V^F$  its Frobenius twist. Considered as a  $G_1$ -module,  $V^F$  is trivial. Conversely, if  $W$  is a  $G$ -module on which  $G_1$  acts trivially then  $W \cong V^F$  for a unique  $G$ -module  $V$  and we write  $W^{(-1)} := V$ .

Consider the Borel subgroup  $B$  of  $G$  consisting of lower triangular matrices and for  $\lambda \in N$ , let  $K_\lambda$  denote the 1-dimensional  $B$ -module of weight  $\lambda$ . Define the *induced*  $G$ -module  $\nabla(\lambda)$  by

$$\nabla(\lambda) := \text{Ind}_B^G(K_\lambda).$$

This is isomorphic to the symmetric power  $S^\lambda E$  where  $E$  is the natural 2-dimensional  $G$ -module. The *Weyl*  $G$ -modules,  $\Delta(\lambda)$ , are defined by

$$\Delta(\lambda) := \nabla(\lambda)^*.$$

Note that  $\text{soc}\nabla(\lambda) = \text{top}\Delta(\lambda) = L(\lambda)$  is simple and  $\{L(\lambda), \lambda \in N\}$  form a complete set of non-isomorphic simple  $G$ -modules. For  $0 \leq \lambda \leq p-1$  we have  $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$  and in general Steinberg's tensor product theorem tells us

that if  $\lambda = \sum_{i \geq 0} \lambda_i p^i$  is the  $p$ -adic expansion of  $\lambda$  then  $L(\lambda)$  is given by

$$L(\lambda) = \bigotimes_{i \geq 0} L(\lambda_i)^{F^i}.$$

The simple  $G$ -modules are thus self-dual.

The modules  $\nabla(\lambda)$  and  $\Delta(\lambda)$  have highest weight  $\lambda$  occuring with multiplicity 1 and all their other weights  $\mu$  satisfy  $\mu < \lambda$ .

In order to prove our results, we use the Lyndon-Hochschild-Serre 5-term exact sequence relating the  $\text{Ext}^1$ -spaces of  $G$  and  $G_1$ . For a rational  $G$ -module  $V$ , we have the exact sequence (see [3])

$$\begin{aligned} 0 \longrightarrow H^1(G, (V^{G_1})^{(-1)}) \longrightarrow H^1(G, V) \longrightarrow H^1(G_1, V)^G \longrightarrow H^2(G, (V^{G_1})^{(-1)}) \\ \longrightarrow H^2(G, V). \end{aligned}$$

In Section 1, we describe properties of  $G_1$ -modules and we compute  $\text{Ext}_{G_1}^i(\Delta, \nabla)$  for  $i \geq 0$  as  $G$ -modules. In Section 2, we use the 5-term exact sequence above and the results of Section 1 to compute  $\text{Ext}_G^1(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t))$  for  $0 \leq k, r$  and  $0 \leq s, t \leq p^n - 1$ . In particular, we show that it has at most dimension 1. We also find explicitly the non-split extensions corresponding to a  $\nabla$ . This filtration of  $\nabla$  by twisted tensor product of  $\nabla$ 's and  $\Delta$ 's explains the symmetries observed in the decomposition matrix of  $G$ .

## 1 Computing $\text{Ext}_{G_1}^i(\Delta, \nabla)$

The category of  $G_1$ -modules is equivalent to the category of  $U$ -modules where  $U$  is the restricted Lie algebra of  $G$ . In particular,  $U$  is a self-injective algebra

(see [15]). This category is very well understood ([9],[14]). The simple  $U$ -modules are the restriction of the  $L(i)$  for  $0 \leq i \leq p-1$  and the corresponding projective  $U$ -modules  $P(i)$  have the following structure: for  $0 \leq i \leq p-2$ ,  $\text{soc}P(i) = \text{top}P(i) = L(i)$  and  $\text{rad}P(i)/\text{soc}P(i) = L(j) \oplus L(j)$  where  $i+j = p-2$  and for  $i = p-1$  the projective module  $P(p-1) = L(p-1)$  is simple. Thus the projective module  $P(p-1)$  is alone in its block and  $P(i)$  and  $P(j)$  belong to the same block if and only if  $i = j$  or  $i+j = p-2$ .

For an indecomposable non-projective  $U$ -module  $M$ , we denote by  $\Omega(M)$  the kernel of the projective cover of  $M$  (and we define inductively  $\Omega^k(M) = \Omega(\Omega^{k-1}(M))$ ). Similarly, we define  $\Omega^{-1}(M)$  to be the cokernel of the injective hull of  $M$  (and we define inductively  $\Omega^{-k}(M)$ ). The projective (injective)  $G_1$ -modules are restrictions of  $G$ -modules and for  $n \geq 0$ , we have an exact sequence of  $G$ -modules ([17], [4])

$$0 \longrightarrow \nabla(np+i) \longrightarrow P(i) \otimes \nabla(n)^F \longrightarrow \nabla((n+1)p+j) \longrightarrow 0.$$

The restriction of this sequence to  $G_1$  gives the projective cover of  $\nabla((n+1)p+j)$  and the injective hull of  $\nabla(np+i)$ .

The  $G_1$ -module  $\nabla(np+i)$  has Loewy length 2 for  $n \geq 1$ . We have a sequence of  $G$ -modules ([17], [12])

$$0 \longrightarrow \nabla(n)^F \otimes \nabla(i) \longrightarrow \nabla(np+i) \longrightarrow \nabla(n-1)^F \otimes \Delta(j) \longrightarrow 0 \quad (1)$$

and its restriction to  $G_1$  gives the Loewy series of  $\nabla(np+i)$  as a  $G_1$ -module.

Note finally that if  $V$ ,  $W$  and  $X$  are  $G$ -modules and  $n \geq 0$  then  $\text{Ext}_{G_1}^n(V, W)$

has a natural structure of  $G$ -module and

$$\mathrm{Ext}_{G_1}^n(V, W \otimes X^F) \cong \mathrm{Ext}_{G_1}^n(V, W) \otimes X^F$$

as  $G$ -modules.

W. van der Kallen proved in [16] that if  $V$  is a  $G$ -module with a good filtration (that is a filtration with quotients isomorphic to some  $\nabla$ 's) then  $H^0(G_1, V)^{(-1)}$  has a good filtration and hence, by dimension shifting (see [7]),  $H^i(G_1, V)^{(-1)}$  has a good filtration for all  $i \geq 0$ . Note that the module  $V = \nabla \otimes \nabla$  has a good filtration and the next two Propositions give the  $G$ -modules  $H^i(G_1, V) = \mathrm{Ext}_{G_1}^i(\Delta, \nabla)$  for  $i \geq 0$ .

Write  $t = t_1p + t_0$  and  $s = s_1p + s_0$  where  $0 \leq s_0, t_0 \leq p - 1$ .

**Proposition 1.1** *For  $i \geq 1$  we have*

$$\mathrm{Ext}_{G_1}^i(\Delta(s), \nabla(t)) \cong \begin{cases} \nabla(s_1 + t_1 + i)^F & \text{if } s_0 + t_0 = p - 2 \text{ and } i \text{ odd} \\ & \text{or } s_0 = t_0 \leq p - 2 \text{ and } i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof:*

From the block structure of  $G_1$  we only need to consider the cases  $s_0 = t_0$  and  $s_0 + t_0 = p - 2$ . Note that if  $s_0 = t_0 = p - 1$  then  $\Delta(s)$  and  $\nabla(t)$  are projective and so there is no non-split extension. Now suppose  $s_0, t_0 \leq p - 2$ .

$$\mathrm{Ext}_{G_1}^i(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \cong \mathrm{Ext}_{G_1}^i(\Omega^{-s_1}(\Delta(s_1p + s_0)), \Omega^{-s_1}(\nabla(t_1p + t_0)))$$

$$\cong \begin{cases} \text{Ext}_{G_1}^i(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) & \text{if } s_1 \text{ even} \\ \text{Ext}_{G_1}^i(\Delta(p - 2 - s_0), \nabla((s_1 + t_1)p + p - 2 - t_0)) & \text{if } s_1 \text{ odd} \end{cases}$$

Now consider the exact sequence,

$$0 \rightarrow \nabla((s_1 + t_1)p + t_0) \rightarrow P(t_0) \otimes \nabla(s_1 + t_1)^F \rightarrow \nabla((s_1 + t_1 + 1)p + p - 2 - t_0) \rightarrow 0$$

and apply  $\text{Hom}_{G_1}(\Delta(s_0), -)$  to get

$$\begin{aligned} 0 \rightarrow \text{Hom}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) &\rightarrow \text{Hom}_{G_1}(\Delta(s_0), P(t_0) \otimes \nabla(s_1 + t_1)^F) \\ &\longrightarrow \text{Hom}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1 + 1)p + p - 2 - t_0)) \\ &\longrightarrow \text{Ext}_{G_1}^1(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \rightarrow 0 \end{aligned} \quad (2)$$

and

$$\text{Ext}_{G_1}^{i+1}(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \cong \text{Ext}_{G_1}^i(\Delta(s_0), \nabla((s_1 + t_1 + 1)p + p - 2 - t_0)).$$

Thus, if we prove the case  $i = 1$  then the result follows by induction. Now, observe that in the exact sequence (2) the first two terms are isomorphic ( $\Delta(s_0)$  is simple and  $P(t_0) \otimes \nabla(s_1 + t_1)^F$  is the injective hull of  $\nabla((s_1 + t_1)p + t_0)$ ), hence the last two terms are isomorphic too and we get

$$\begin{aligned} &\text{Ext}_{G_1}^1(\Delta(s_0), \nabla((s_1 + t_1)p + t_0)) \\ &\cong \text{Hom}_{G_1}(\Delta(s_0), \nabla((s_1 + t_1 + 1)p + p - 2 - t_0)) \\ &\cong \text{Hom}_{G_1}(\Delta(s_0), P(p - 2 - t_0) \otimes \nabla(s_1 + t_1 + 1)^F) \\ &\cong \text{Hom}_{G_1}(\Delta(s_0), P(p - 2 - t_0) \otimes \nabla(s_1 + t_1 + 1)^F) \\ &\cong \begin{cases} \nabla(s_1 + t_1 + 1)^F & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$



The proposition then follows by induction on  $i$ .

QED

**Proposition 1.2**

$$\mathrm{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof:*

Note that by the decomposition into blocks of  $G_1$ , we only need to consider the cases  $s_0 + t_0 = p - 2$  and  $s_0 = t_0$ . Suppose for a start that  $s_0, t_0 \leq p - 2$ .

Consider the exact sequence

$$0 \longrightarrow \nabla(t_1)^F \otimes \nabla(t_0) \longrightarrow \nabla(t_1p + t_0) \longrightarrow \nabla(t_1 - 1)^F \otimes \Delta(p - 2 - t_0) \longrightarrow 0.$$

Apply  $\mathrm{Hom}_{G_1}(\Delta(s_1p + s_0), -)$  to get the exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) &\rightarrow \mathrm{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \\ \rightarrow \mathrm{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1 - 1)^F \otimes \Delta(p - 2 - t_0)) &\rightarrow \mathrm{Ext}_{G_1}^1(\Delta(s_1p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) \\ &\rightarrow \mathrm{Ext}_{G_1}^1(\Delta(s_1p + s_0), \nabla(t_1p + t_0)). \end{aligned} \tag{3}$$

Now,

$$\begin{aligned} \mathrm{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) &\cong \mathrm{Hom}_{G_1}(\nabla(t_0), \nabla(s_1p + s_0)) \otimes \nabla(t_1)^F \\ &\cong \mathrm{Hom}_{G_1}(\nabla(t_0), P(s_0)) \otimes \nabla(s_1)^F \otimes \nabla(t_1)^F \\ &\cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F & \text{if } s_0 = t_0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\begin{aligned}
& \text{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1 - 1)^F \otimes \Delta(p - 2 - t_0)) \\
& \cong \text{Hom}_{G_1}(\nabla(p - 2 - t_0), \nabla(s_1p + s_0)) \otimes \nabla(t_1 - 1)^F \\
& \cong \text{Hom}_{G_1}(\nabla(p - 2 - t_0), P(s_0)) \otimes \nabla(s_1)^F \otimes \nabla(t_1 - 1)^F \\
& \cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1 - 1))^F & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Using Proposition 1.1, we get

$$\begin{aligned}
& \text{Ext}_{G_1}^1(\Delta(s_1p + s_0), \nabla(t_1)^F \otimes \nabla(t_0)) \cong \text{Ext}_{G_1}^1(\nabla(t_0), \nabla(s_1p + s_0)) \otimes \nabla(t_1)^F \\
& \cong \begin{cases} (\nabla(s_1 + 1) \otimes \nabla(t_1))^F & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and

$$\text{Ext}_{G_1}^1(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \cong \begin{cases} \nabla(s_1 + t_1 + 1)^F & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise.} \end{cases}$$

So if  $s_0 + t_0 = p - 2$  and  $p > 2$  (i.e.  $s_0 \neq t_0$ ), then the exact sequence (3)

becomes

$$\begin{aligned}
0 & \longrightarrow \text{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) \longrightarrow (\nabla(s_1) \otimes \nabla(t_1 - 1))^F \\
& \longrightarrow (\nabla(s_1 + 1) \otimes \nabla(t_1))^F \longrightarrow \nabla(s_1 + t_1 + 1)^F.
\end{aligned}$$

As

$$\dim(\nabla(s_1 + 1) \otimes \nabla(t_1))^F = \dim(\nabla(s_1) \otimes \nabla(t_1 - 1))^F + \dim \nabla(s_1 + t_1 + 1)^F,$$

we deduce that

$$\text{Hom}_{G_1}(\Delta(s_1p + s_0), \nabla(t_1p + t_0)) = 0.$$

If  $s_0 = t_0$  and  $p = 2$ , the exact sequence (3) has the form

$$\begin{aligned} 0 &\longrightarrow (\nabla(s_1) \otimes \nabla(t_1))^F \longrightarrow \text{Hom}_{G_1}(\Delta(s_1 2 + s_0), \nabla(t_1 2 + t_0)) \\ &\longrightarrow (\nabla(s_1) \otimes \nabla(t_1 - 1))^F \longrightarrow (\nabla(s_1 + 1) \otimes \nabla(t_1))^F \longrightarrow \nabla(s_1 + t_1 + 1)^F. \end{aligned}$$

Hence,

$$\text{Hom}_{G_1}(\Delta(s_1 2 + s_0), \nabla(t_1 2 + t_0)) \cong (\nabla(s_1) \otimes \nabla(t_1))^F.$$

Finally if  $s_0 = t_0$  and  $p > 2$  then clearly

$$\text{Hom}_{G_1}(\Delta(s_1 p + s_0), \nabla(t_1 p + t_0)) \cong (\nabla(s_1) \otimes \nabla(t_1))^F.$$

In the case where  $s_0 = t_0 = p - 1$ , we have the following

$$\begin{aligned} \Delta(s_1 p + s_0) &\cong \Delta(s_1)^F \otimes \Delta(p - 1) \\ \nabla(t_1 p + t_0) &\cong \nabla(t_1)^F \otimes \nabla(p - 1), \end{aligned}$$

and so

$$\begin{aligned} &\text{Hom}_{G_1}(\Delta(s_1 p + (p - 1)), \nabla(t_1 p + (p - 1))) \\ &\cong \text{Hom}_{G_1}(\Delta(p - 1), \nabla(p - 1)) \otimes (\nabla(s_1) \otimes \nabla(t_1))^F \\ &\cong (\nabla(s_1) \otimes \nabla(t_1))^F. \end{aligned}$$

This completes the proof.

QED

## 2 Extensions of $G$ -modules

In [5] and [8], Cox and Erdmann determined the  $\text{Ext}^1$  and the Hom spaces between  $\nabla(\lambda)$  and  $\nabla(\mu)$  for arbitrary weights  $\lambda$  and  $\mu$ . For completeness and

to fix our notation, we state their result here.

For  $0 \leq a \leq p-1$  denote by  $\hat{a}$ , the integer such that  $a + \hat{a} = p-1$ . For a weight  $\mu$ , define

$$\psi^0(\mu) = \left\{ \sum_{i=0}^{u-1} \hat{\mu}_i p^i : u \geq 0 \right\}$$

and

$$\psi^1(\mu) = \left\{ \sum_{i=0}^{u-1} \hat{\mu}_i p^i + p^{u+a} : \hat{\mu}_u \neq 0, a \geq 1, u \geq 0 \right\} \cup \left\{ \sum_{i=0}^u \hat{\mu}_i p^i : \hat{\mu}_u \neq 0, u \geq 0 \right\}.$$

With this notation we have,

$$\text{Hom}_G(\nabla(\lambda), \nabla(\mu)) \cong \begin{cases} K & \text{if } \lambda = \mu + 2d, d \in \psi^0(\mu) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

and

$$\text{Ext}_G^1(\nabla(\lambda), \nabla(\mu)) \cong \begin{cases} K & \text{if } \lambda = \mu + 2e, e \in \psi^1(\mu) \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

In [2], Cline determined all the  $\text{Ext}^1$ -spaces between simple  $G$ -modules. In particular, for simple modules  $\nabla(r)^F \otimes \nabla(s)$  and  $\nabla(k)^F \otimes \nabla(t)$ , he proved that

$$\text{Ext}_G^1(\nabla(r)^F \otimes \nabla(s), \nabla(k)^F \otimes \nabla(t)) \cong \begin{cases} K & \text{if } r = k \pm 1, s + t = p - 2 \\ 0 & \text{otherwise} \end{cases}$$

The following theorem extends this result.

**Theorem 2.1** *Let  $0 \leq k, r$  and  $0 \leq s, t \leq p^n - 1$  then we have*

$$\text{Ext}_G^1(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t)) \cong \begin{cases} K & \text{if } \begin{aligned} &r = k + 2e, \ e \in \psi^1(k) \\ &s = t \end{aligned} \\ \\ \\ \\ \\ 0 & \text{otherwise} \end{cases}$$

*Proof:*

In order to prove this theorem, we use the five terms exact sequence:

$$0 \longrightarrow H^1(G, (V^{G_1})^{(-1)}) \longrightarrow H^1(G, V) \longrightarrow H^1(G_1, V)^G \longrightarrow H^2(G, (V^{G_1})^{(-1)}) \longrightarrow H^2(G, V),$$

with  $V = \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \otimes \nabla(s) \otimes \nabla(t)$ .

Write  $s = s_1p + s_0$  and  $t = t_1p + t_0$ . Let us first compute  $H^1(G, (V^{G_1})^{(-1)})$ .

Using Proposition 1.2, we have

$$\begin{aligned} V^{G_1} &= \text{Hom}_{G_1}(\Delta(s), \nabla(t)) \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \\ &\cong \begin{cases} (\nabla(s_1) \otimes \nabla(t_1))^F \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now,

$$(V^{G_1})^{(-1)} \cong \begin{cases} \nabla(s_1) \otimes \nabla(t_1) \otimes \Delta(r)^{F^{n-1}} \otimes \nabla(k)^{F^{n-1}} & \text{if } s_0 = t_0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence for  $s_0 = t_0$  we have

$$H^1(G, (V^{G_1})^{(-1)}) \cong \text{Ext}_G^1(\nabla(r)^{F^{n-1}} \otimes \Delta(s_1), \nabla(k)^{F^{n-1}} \otimes \nabla(t_1)),$$

and is zero in all other cases.

Let us now compute  $H^1(G_1, V)^G$ . Using Proposition 1.1, we have

$$\begin{aligned} H^1(G_1, V) &= \text{Ext}_{G_1}^1(\nabla(r)^{F^n} \otimes \Delta(s), \nabla(k)^{F^n} \otimes \nabla(t)) \\ &\cong \text{Ext}_{G_1}^1(\Delta(s), \nabla(t)) \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} \\ &\cong \begin{cases} \nabla(s_1 + t_1 + 1)^F \otimes \Delta(r)^{F^n} \otimes \nabla(k)^{F^n} & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$H^1(G_1, V)^G \cong \begin{cases} \text{Hom}_G(\Delta(s_1 + t_1 + 1)^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}) & \text{if } s_0 + t_0 = p - 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that all the weights of  $\Delta(r)^{F^n} \otimes \nabla(k)^{F^n}$  are multiples of  $p^n$ , so to get non-zero homomorphisms, we must have  $s_1 + t_1 + 1 = cp^{n-1}$  for some  $c$ . But  $s, t \leq p^n - 1$  implies that  $s_1 + t_1 \leq 2p^{n-1} - 2$ , thus  $c = 1$  and  $s_1 + t_1 + 1 = p^{n-1}$ .

Observe that

$$\text{Hom}_G(\Delta(p^{n-1})^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}) \cong \text{Hom}_G(\nabla(r)^{F^n}, \nabla(p^{n-1})^F \otimes \nabla(k)^{F^n})$$

and that all the weights of  $\nabla(r)^{F^n}$  are multiple of  $p^n$  so the image of a homomorphism from  $\nabla(r)^{F^n}$  to  $\nabla(p^{n-1})^F \otimes \nabla(k)^{F^n}$  lies in the submodule

$$\nabla(1)^{F^n} \otimes \nabla(k)^{F^n} \leq \nabla(p^{n-1})^F \otimes \nabla(k)^{F^n}. \text{ Hence,}$$

$$\begin{aligned} \text{Hom}_G(\Delta(p^{n-1})^F, \Delta(r)^{F^n} \otimes \nabla(k)^{F^n}) &\cong \text{Hom}_G(\nabla(r)^{F^n}, \nabla(1)^{F^n} \otimes \nabla(k)^{F^n}) \\ &\cong \text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)). \end{aligned}$$

We claim that  $\text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) \cong K$  if  $r = k \pm 1 + 2d$  where  $d \in \psi^0(k)$  and zero otherwise. Consider the exact sequence

$$0 \longrightarrow \nabla(z-1) \longrightarrow \nabla(1) \otimes \nabla(z) \longrightarrow \nabla(z+1) \longrightarrow 0. \quad (6)$$

This sequence splits if and only if  $z \not\equiv -1 \pmod{p}$ . Note that for  $\text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k))$  to be non zero, we must have  $r+k \equiv 1 \pmod{2}$ . Now suppose  $k \equiv -1 \pmod{p}$  then we can assume  $r \not\equiv -1 \pmod{p}$  and so using (6) with  $z = r$  we have

$$\begin{aligned} \text{Hom}_G(\nabla(r), \nabla(1) \otimes \nabla(k)) &\cong \text{Hom}_G(\nabla(1) \otimes \nabla(r), \nabla(k)) \\ &\cong \text{Hom}_G(\nabla(r-1) \oplus \nabla(r+1), \nabla(k)). \end{aligned}$$

Now, using (4) we deduce that  $\text{Hom}_G(\nabla(r-1), \nabla(k)) \cong K$  if and only if  $r-1 = k+2d$  where  $d \in \psi^0(k)$  and it is zero otherwise, and  $\text{Hom}_G(\nabla(r+1), \nabla(k)) \cong K$  if and only if  $r+1 = k+2d'$  where  $d' \in \psi^0(k)$  and zero otherwise. Suppose they are both non-zero then  $k+1+2d = k-1+2d'$ . But this can only happen when  $d = 0$ ,  $d' = 1$  and  $r = k+1$ . This means that  $k \equiv p-2 \pmod{p}$  and  $r \equiv -1 \pmod{p}$  contradicting our assumption. Now if  $k \not\equiv -1 \pmod{p}$  we use (6) with  $z = k$  and the claim follows by a similar argument.

Hence, we have proved the following

$$H^1(G_1, V)^G \cong \begin{cases} K & \text{if } s_0 + t_0 = p-2, s_1 + t_1 = p^{n-1} - 1 \\ & r = k \pm 1 + 2d \text{ where } d \in \psi^0(k) \\ 0 & \text{otherwise.} \end{cases}$$

Let us now use the five term sequence to determine  $H^1(G, V)$ . We shall do this by induction on  $n$ . For  $n = 1$  we have  $s, t \leq p - 1$  and

$$H^1(G, (V^{G_1})^{(-1)}) \cong \begin{cases} K & \text{if } r = k + 2e, e \in \psi^1(k) \text{ and } s = t \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^1(G_1, V)^G \cong \begin{cases} K & \text{if } r = k \pm 1 + 2d, d \in \psi^0(k) \text{ and } s + t = p - 2 \\ 0 & \text{otherwise,} \end{cases}$$

thus,

$$H^1(G, V) \cong \begin{cases} K & \text{if } r = k + 2e, e \in \psi^1(k) \text{ and } s = t \\ & \text{or } r = k \pm 1 + 2d, d \in \psi^0(k) \text{ and } s + t = p - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now we use induction. Note that if  $p = 2$  and  $s_0 = t_0 = 0$  and  $s_1 + t_1 = 2^{n-1} - 1$  then  $\Delta(s_1)$  and  $\nabla(t_1)$  are in different blocks of  $G_1$  and so

$$\text{Ext}_G^i(\nabla(r)^{F^{n-1}} \otimes \Delta(s_1), \nabla(k)^{F^{n-1}} \otimes \nabla(t_1)) = 0 \quad \text{for all } i.$$



So for all prime  $p$  we get

$$H^1(G, V) \cong \begin{cases} K & \text{if } \begin{aligned} r &= k + 2e, \ e \in \psi^1(k) \\ s &= t \end{aligned} \\ \\ r &= k \pm 1 + 2d, \ d \in \psi^0(k) \\ \text{or } t &= t_0 + t_1 p^i, \ 0 \leq t_0 \leq p^i - 1 \\ s &= t_0 + (p^{n-i} - 2 - t_1) p^i \\ \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of our theorem.

QED

Note that if we set  $n = 0$  and  $s = t = 0$  in Theorem 2.1 we get Erdmann and Cox's result given by equation (5).

The following proposition shows that when  $r = k - 1$  and  $s = p^n - 2 - t$ , the extension is given by  $\nabla(kp^n + t)$ . By considering weights, it is easy to see that no other extension described in Theorem 2.1 can be isomorphic to an induced module  $\nabla(\lambda)$ .

**Proposition 2.1** *For  $k \in N$  and  $0 \leq t \leq p^n - 2$ , there is an exact sequence of  $G$ -modules*

$$0 \longrightarrow \nabla(k)^{F^n} \otimes \nabla(t) \longrightarrow \nabla(kp^n + t) \longrightarrow \nabla(k - 1)^{F^n} \otimes \Delta(p^n - 2 - t) \longrightarrow 0.$$

*Moreover,  $\nabla(kp^n + t)$  is the only non-split extension, up to isomorphism, of  $\nabla(k - 1)^{F^n} \otimes \Delta(p^n - t - 2)$  by  $\nabla(k)^{F^n} \otimes \nabla(t)$ .*

Dually, the only non-split extension, up to isomorphism, of  $\Delta(k)^{F^n} \otimes \Delta(t)$  by  $\Delta(k-1)^{F^n} \otimes \nabla(p^n - t - 2)$  is given by  $\Delta(kp^n + t)$ .

**Remark 1:** For  $k \in N$  we have an isomorphism between  $\nabla(k-1)^{F^n} \otimes St_n$  and  $\nabla(kp^n - 1)$  given by multiplication of polynomials. It is known that there is an isomorphism between these modules more generally, see for example [11](II.3).

*Proof of Proposition 2.1:*

If  $n = 1$  then we are done by (1) (Section 1). Suppose  $n > 1$  and write  $t = ap^{n-1} + d$ , for  $0 \leq a \leq p-1$  and  $0 \leq d \leq p^{n-1} - 1$ . Using induction we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \nabla(kp+a)^{F^{n-1}} \otimes \nabla(d) &\longrightarrow \nabla((kp+a)p^{n-1} + d) \longrightarrow \\ &\nabla(kp+(a-1))^{F^{n-1}} \otimes \Delta(p^{n-1} - d - 2) \longrightarrow 0. \end{aligned}$$

Using the exact sequences (1) for  $\nabla(kp+a)^{F^{n-1}}$  and  $\nabla(kp+(a-1))^{F^{n-1}}$  we get a filtration of  $\nabla(kp^n + ap^{n-1} + d)$  with quotients

$$\begin{aligned} &\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1} - d - 2) \\ &\nabla(k)^{F^n} \otimes \nabla(a-1)^{F^{n-1}} \otimes \Delta(p^{n-1} - d - 2) \\ &\nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d) \\ &\nabla(k)^{F^n} \otimes \nabla(a)^{F^{n-1}} \otimes \nabla(d) \end{aligned}$$

Observe that the module  $\nabla(kp^n + ap^{n-1} + d)$  is multiplicity-free, so that the four quotients have disjoint sets of weights. Hence,  $\nabla(kp^n + t)/\nabla(k)^{F^n} \otimes \nabla(t)$

has a filtration with quotients

$$\begin{aligned} & \nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1}-d-2) \\ & \nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d) \end{aligned}$$

Note that for  $a = p-1$  or  $d = p^{n-1}-1$ , we only have one factor appearing and so we are done by Remark 1 above. So suppose  $a \leq p-2$  and  $d \leq p^{n-1}-2$ . Using a very similar argument to the proof of Theorem 2.1 we can show that

$$\begin{aligned} & \text{Ext}_G^1(\nabla(k-1)^{F^n} \otimes \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1}-d-2), \\ & \nabla(k-1)^{F^n} \otimes \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d)) \cong K. \end{aligned}$$

Now as  $\nabla(kp^n+t)$  has simple top (see [1]),  $\nabla(kp^n+t)/\nabla(k)^{F^n} \otimes \nabla(t)$  cannot be a direct sum of non-zero modules. By induction, we know that  $\Delta(p^n - ap^{n-1} - d - 2)$  has a filtration with quotients

$$\begin{aligned} & \Delta(p-a-1)^{F^{n-1}} \otimes \Delta(p^{n-1}-d-2) \\ & \Delta(p-a-2)^{F^{n-1}} \otimes \nabla(d) \end{aligned}$$

We deduce that the quotient  $\nabla(kp^n+t)/\nabla(k)^{F^n} \otimes \nabla(t)$  is isomorphic to

$$\nabla(k-1)^{F^n} \otimes \Delta(p^n - ap^{n-1} - d - 2) = \nabla(k-1)^{F^n} \otimes \Delta(p^n - 2 - t).$$

This completes the proof

QED

**Remark 2:** S.Donkin suggested an alternative proof of Proposition 2.1. I shall sketch his argument here. Let us start with the exact sequence of  $B$ -modules

$$0 \longrightarrow \nabla(s-1) \otimes K_{-1} \longrightarrow \nabla(s) \longrightarrow K_s \longrightarrow 0 \quad (7)$$

for any positive integer  $s$ . Apply the Frobenius morphism  $F^n$  to the sequence (7) and tensor it with  $K_r$  for some  $0 \leq r \leq p^n - 1$ . Then applying the induction functor from  $B$ -modules to  $G$ -modules and using the duality of induction (see [11], II.4) gives the required sequence.

**Remark 3:** The composition factors of the  $\nabla$ 's are known for  $SL(2, K)$  (use for example equation (1) repeatedly) but Proposition 2.1 gives a direct explanation of the symmetries observed by A. Henke in the decomposition matrix of  $SL(2, K)$  (see [10]). More precisely, if we write  $\lambda = kp^n + t$  with  $k \leq p - 1$  then our proposition tells us that

$$[\nabla(kp^n + t) : L(kp^n + a)] = [\nabla(t) : L(a)],$$

$$[\nabla(kp^n + t) : L((k - 1)p^n + b)] = [\nabla(p^n - 2 - t) : L(b)].$$

Let us write the decomposition matrix of  $G$  with the  $\nabla$ 's on the horizontal axis and the  $L$ 's on the vertical axis (see figures 1 and 2 below). Then for each  $n \geq 1$  and each  $1 \leq k \leq p - 1$ , the columns corresponding to  $\nabla(kp^n + t)$  for  $0 \leq t \leq p^n - 1$  are obtained from the left bottom  $p^n \times p^n$  block by

1. Translation of length  $k$  along the diagonal,
2. Translation of length  $k - 1$  along the diagonal and then reflection through the column corresponding to  $\nabla(kp^n - 1)$  (shaded on the figures).

Hence, we can construct the decomposition matrix inductively starting with the left bottom  $p \times p$  block which is just a diagonal matrix, as for  $0 \leq r \leq p - 1$  we have  $\nabla(r) = L(r)$ .

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